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A NOTE ON DUALITY IN DISJUNCTIVE PROGRAMMING.(U)

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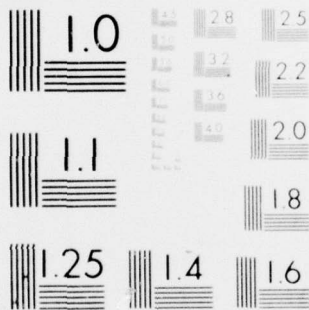
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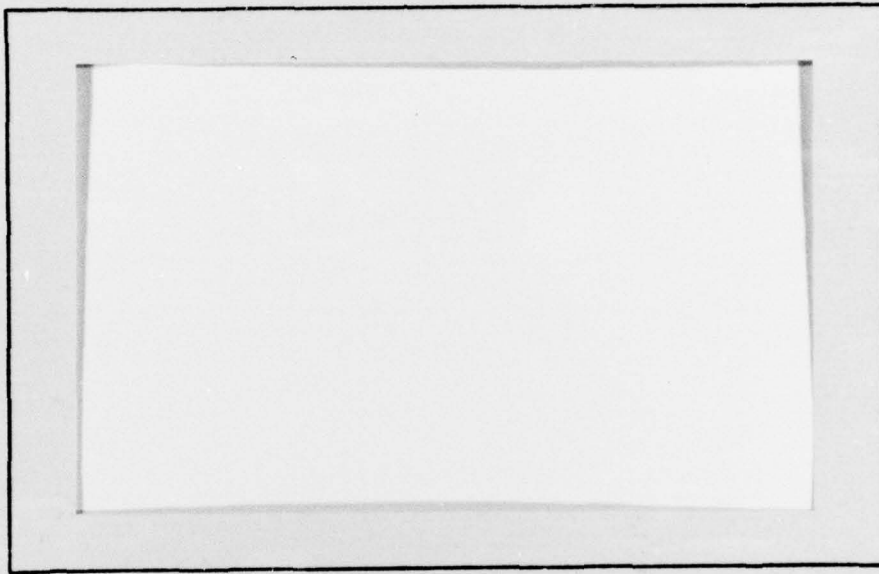
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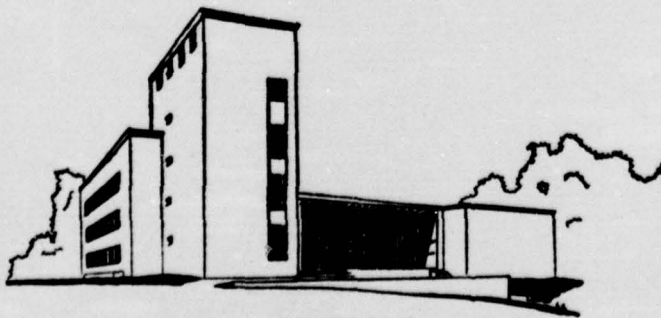


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by

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Egon Balas

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Abstract

We state a duality theorem for disjunctive programming, which generalizes to this class of problems the corresponding result for linear programming.

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A NOTE ON  
DUALITY IN DISJUNCTIVE PROGRAMMING

by

Egon Balas

1. Introduction.

A disjunctive programming problem is a linear programming problem with disjunctive constraints. Mixed integer programs and many other nonconvex programming problems can be stated in this form. One advantage of this formulation is that it yields a variety of cutting planes with desirable properties (see Refs. 1,...,4; and, for an early version, 5). Another one is that it leads to nice theoretical characterizations: a disjunctive program can be shown to be equivalent to a linear program; the family of all valid inequalities can be described in terms of a scaled polar set; the facets of the convex hull of feasible points are the extreme points or extreme half-lines of this polar set; for a large family of disjunctive programs (which includes the zero-one mixed integer program, the linear complementarity problem and other familiar models), the convex hull of feasible points can be generated sequentially, imposing the disjunctions of a conjunctive normal form one by one (see Ref. 6).

In this note we state a duality theorem for disjunctive programs, which generalizes to this class of problems the duality theorem of linear programming.



## 2. The Duality Theorem.

Consider the disjunctive program

$$z_0 = \min cx$$

$$(P) \quad \bigvee_{h \in Q} \left\{ \begin{array}{l} A^h x \geq b^h \\ x \geq 0 \end{array} \right\},$$

where  $A^h$  is a matrix and  $b^h$  a vector,  $\forall h \in Q$ . The constraint set of (P) requires  $x$  to satisfy at least one of the  $|Q|$  bracketed systems of inequalities.

We define the dual of (P) to be the problem

$$w_0 = \max w$$

$$(D) \quad \bigwedge_{h \in Q} \left\{ \begin{array}{l} w - u^h b^h \leq 0 \\ u^h A^h \leq c \\ u^h \geq 0 \end{array} \right\}$$

The constraint set of (D) requires each  $u^h$ ,  $h \in Q$ , to satisfy the corresponding bracketed system, and  $w$  to satisfy each of them.

Let

$$x_h = \{x | A^h x \geq b^h, x \geq 0\}, \quad \bar{x}_h = \{x | A^h x \geq 0, x \geq 0\};$$

$$u_h = \{u^h | u^h A^h \leq c, u^h \geq 0\}, \quad \bar{u}_h = \{u^h | u^h A^h \leq 0, u^h \geq 0\}.$$

Further, let

$$Q^* = \{h \in Q | x_h \neq \emptyset\}, \quad Q^{**} = \{h \in Q | u_h \neq \emptyset\}.$$

We will assume the following

Regularity condition:

$$(Q^* \neq \emptyset, Q \setminus Q^{**} \neq \emptyset) \Rightarrow Q^* \setminus Q^{**} \neq \emptyset;$$

i.e., if (P) is feasible and (D) is infeasible, then there exists  $h \in Q$  such that  $X_h \neq \emptyset, U_h = \emptyset$ .

Theorem. Assume that (P) and (D) satisfy the regularity condition.

Then exactly one of the following two situations holds.

1. Both problems are feasible; each has an optimal solution and  $z_0 = w_0$ .
2. One of the problems is infeasible; the other one either is infeasible or has no finite optimum.

Proof. (1) Assume that both (P) and (D) are feasible. If (P) has no finite minimum, then there exists  $h \in Q$  such that  $\bar{X}_h \neq \emptyset$  and  $\bar{x} \in \bar{X}_h$  such that  $c\bar{x} < 0$ . But then  $U_h = \emptyset$ , i.e., (D) is infeasible; a contradiction.

Thus (P) has an optimal solution, say  $\bar{x}$ . Then the inequality  $cx \geq z_0$  is a consequence of the constraint set of (P); i.e.,  $x \in X_h$  implies  $cx \geq z_0$ ,  $\forall h \in Q$ . But then for all  $h \in Q^*$ , there exists  $u^h \in U_h$  such that  $u^h b^h \geq z_0$ . Further, since (D) is feasible, for each  $h \in Q \setminus Q^*$  there exists  $\hat{u}^h \in U_h$ ; and since  $X_h \neq \emptyset$  (for  $h \in Q \setminus Q^*$ ), there also exists  $\bar{u}^h \in \bar{U}_h$  such that  $\bar{u}^h b^h > 0$ ,  $\forall h \in Q \setminus Q^*$ . But then, defining

$$u^h(\lambda) = \hat{u}^h + \lambda \bar{u}^h, \quad h \in Q \setminus Q^*,$$

for  $\lambda$  sufficiently large,  $u^h(\lambda) \in U_h$ ,  $u^h(\lambda) b^h \geq z_0$ ,  $\forall h \in Q \setminus Q^*$ .

Hence for all  $h \in Q$ , there exist vectors  $u^h$  satisfying the constraints of (D) for  $w = z_0$ . To show that this is the maximal value of  $w$ , we note that since  $\bar{x}$  is optimal for (P), there exists  $h \in Q$  such that



$$c\bar{x} = \min\{c\bar{x} \mid \bar{x} \in X_h\}.$$

But then by linear programming duality,

$$\begin{aligned} c\bar{x} &= \max\{u^h b^h \mid u^h \in U_h\} \\ &= \max\{w \mid w - u^h b^h \leq 0, u^h \in U_h\} \\ &\geq \max_{h \in Q}\{w \mid (w - u^h b^h \leq 0, u^h \in U_h)\} \end{aligned}$$

i.e.,  $w \leq z_0$ , and hence the maximum value of  $w$  is  $w_0 = z_0$ .

(ii) Assume that at least one of (P) and (D) is infeasible. If (P) is infeasible,  $X_h = \emptyset$ ,  $\forall h \in Q$ ; hence for all  $h \in Q$ , there exists  $\bar{u}^h \in \bar{U}_h$  such that  $\bar{u}^h b^h > 0$ .

If (D) is infeasible, we are done. Otherwise, for each  $h \in Q$  there exists  $\hat{u}^h \in U_h$ . But then defining

$$u^h(\lambda) = \hat{u}^h + \lambda \bar{u}^h, \quad h \in Q,$$

$u^h(\lambda) \in U_h$ ,  $h \in Q$ , for all  $\lambda > 0$ , and since  $\bar{u}^h b^h > 0$ ,  $\forall h \in Q$ ,  $w$  can be made arbitrarily large by increasing  $\lambda$ ; i.e., (D) has no finite optimum.

Conversely, if (D) is infeasible, then either (P) is infeasible and we are done, or else, from the regularity condition,  $Q^* \setminus Q^{**} \neq \emptyset$ ; and for  $h \in Q^* \setminus Q^{**}$  there exists  $\hat{x} \in X_h$  and  $\bar{x} \in \bar{X}_h$  such that  $c\bar{x} < 0$ . But then

$$x(\mu) = \hat{x} + \mu \bar{x}$$

is a feasible solution to (P) for any  $\mu > 0$ , and since  $c\bar{x} < 0$ ,  $z$  can be made arbitrarily small by increasing  $\mu$ ; i.e., (P) has no finite optimum.

Q.E.D.

### 3. Discussion.

The above theorem asserts that either situation 1 or situation 2 holds for (P) and (D) if the regularity condition is satisfied. The following Corollary shows that the condition is not only sufficient but also necessary.

Corollary: If the regularity condition does not hold, then if (P) is feasible and (D) is infeasible, (P) has a finite minimum (i.e., there is a "duality gap").

Proof. Let (P) be feasible, (D) infeasible, and  $Q^* \setminus Q^{**} = \emptyset$ , i.e., for every  $h \in Q^*$ , let  $U_h \neq \emptyset$ . Then for each  $h \in Q^*$ ,  $\min\{cx \mid x \in X_h\}$  is finite, hence (P) has a finite minimum. Q.E.D.

Remark. The theorem remains true if some of the variables of (P) [of (D)] are unconstrained, and the corresponding constraints of (D) [of (P)] are equalities.

The regularity condition can be expected to hold in all but some rather peculiar situations. In linear programming duality, the case when both the primal and the dual problem is infeasible only occurs for problems whose coefficient matrix A has the rather special property that there exists  $x \neq 0$ ,  $u \neq 0$ , satisfying the homogeneous system

$$Ax \geq 0, \quad x \geq 0$$

$$uA \leq 0, \quad u \geq 0$$

(see reference 7 for a discussion of this and some equivalent conditions). In this context, our regularity condition requires that, if the primal problem is feasible and the dual is infeasible, then at least one of the matrices  $A^h$  whose associated sets  $U_h$  are infeasible, should not have the above mentioned special property.

Though most problems satisfy this requirement, nevertheless there are situations when the regularity condition breaks down, as illustrated by the following example.

Consider the disjunctive program

$$\min -x_1 - 2x_2$$

$$(P) \quad \left\{ \begin{array}{l} -x_1 + x_2 \geq 0 \\ -x_1 - x_2 \geq -2 \\ x_1, x_2 \geq 0 \end{array} \right\} \vee \left\{ \begin{array}{l} -x_1 + x_2 \geq 0 \\ x_1 - x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{array} \right\}$$

and its dual

$$(D) \quad \begin{array}{ll} \max w & \\ w + 2u_2^1 & \leq 0 \\ -u_1^1 - u_2^1 & \leq -1 \\ u_1^1 - u_2^1 & \leq -2 \\ w - u_2^2 & \leq 0 \\ -u_1^2 + u_2^2 & \leq -1 \\ u_1^2 - u_2^2 & \leq -2 \\ u_i^k \geq 0, & i = 1, 2; k = 1, 2 \end{array}$$

The primal problem (P) has an optimal solution  $\bar{x} = (0, 2)$ , with  $c\bar{x} = -4$ ; whereas the dual problem (D) is infeasible. This is due to the fact that  $Q \setminus Q^{**} = \{2\}$  and  $X_2 = \emptyset$ ,  $U_2 = \emptyset$ , i.e., the regularity condition is violated.

Here

$$X_2 = \left\{ x \in R_+^2 \mid \begin{array}{l} -x_1 + x_2 \geq 0 \\ x_1 - x_2 \geq 1 \end{array} \right\}, \quad U_2 = \left\{ u \in R_+^2 \mid \begin{array}{l} -u_1^2 + u_2^2 \leq -1 \\ u_1^2 - u_2^2 \leq -2 \end{array} \right\}.$$

The duality theorem discussed in this paper can be used to derive strong lower bounds on the value of  $z_0$  in the context of a branch and bound procedure. This is best done by using an appropriate relaxation of (P). For instance, in the case of a zero-one program the relaxation may consist of considering only one or two of the disjunctions  $x_j = 0$  or  $1$ , the locally most relevant ones. In that case the linear program (D) is of manageable size; furthermore, (D) need not be solved completely, any feasible solution to it provides a valid lower bound on the value of  $z_0$ .

Another potential use is the derivation of strong cutting planes. If  $(\bar{w}, \bar{u})$  is any feasible solution to (D) and  $a_j^h, j \in N$ , are the columns of  $A^h, h \in Q$ , then

$$\sum_{j \in N} (\max_{h \in Q^*} \bar{u}^h a_j^h) x_j \geq \bar{w}$$

is a valid cutting plane for (P) (see ref. 2).



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